

Sticky grains do not change the universality class of isotropic sandpiles

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We revisit the sandpile model with “sticky” grains introduced by Mohanty and Dhar [Phys. Rev. Lett. **89**, 104303 (2002)] whose scaling properties were claimed to be in the universality class of directed percolation for both isotropic and directed models. Simulations in the so-called fixed-energy ensemble show that this conclusion is not valid for isotropic sandpiles and that this model shares the same critical properties of other stochastic sandpiles, such as the Manna model. These results are strengthened by the analysis of the Langevin equations proposed by the same authors to account for this problem which we show to converge, upon coarse-graining, to the well-established set of Langevin equations for the Manna class. Therefore, the presence of a conservation law keeps isotropic sandpiles, with or without stickiness, away from the directed percolation class.

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I. INTRODUCTION

Toy models of sandpiles are the archetypical examples of self-organized criticality emerging out of time-scale separation [1, 2, 3]. Sandpile models come in many different flavors (deterministic or stochastic rules [4], discrete or continuous variables [5], with or without height restrictions [6], etc.), but they usually consist in adding grains one by one until a local threshold (typically a condition on some slope or height) is reached, triggering a series of redistribution events, i.e. “avalanches”, which may lead to dissipation of sandgrains at the open boundaries. Their numerical study is notoriously difficult and first led to a largely unsatisfactory situation in which “microscopic details” were believed to influence scaling properties, in contradiction with universality principles [7, 8]. Major progress in favor of universality came when sandpile criticality was put into the broader context of standard non-equilibrium absorbing-state phase transitions [9, 10, 11, 12] [13, 14, 15, 16, 17, 25].

Indeed, switching off both dissipation (open boundaries) and driving (slow addition of grains), the total amount of sand or “energy” is conserved, and becomes a control parameter for these “fixed energy sandpiles”. For large amounts of sand, the system is in an active phase with never-ending relaxation events, while for small energies it gets trapped with certainty into some absorbing state where all dynamics ceases (all sites being below threshold). Separating these two regimes there is a critical energy which was shown [13, 14, 15] to coincide with the stationary energy-density in the corresponding original sandpile (with slow-driving and boundary dissipation). In this way, the exponents characterizing sandpiles can be related to standard critical exponents in an absorbing-state phase transition [18] (an alternative route not discussed here is to map sandpiles into standard pinning-depinning interfacial phase transitions).

Using this approach, it was determined that stochas-

tic sandpiles [20] generally do *not* belong to the directed percolation (DP) class, prominent among absorbing phase transitions, but to the so-called “conserved-DP” or Manna class (hereafter C-DP/Manna) characterized by the coupling of activity to a static conserved field directly representing the conservation of sandgrains [13, 14, 15, 17, 21, 22]. The field theory or mesoscopic Langevin equations describing this class reads:

$$\begin{aligned}\partial_t \rho &= a\rho - b\rho^2 + \omega\rho E + D\nabla^2 \rho + \sigma\sqrt{\rho}\eta \\ \partial_t E &= D_E \nabla^2 \rho\end{aligned}\quad (1)$$

where $\rho(\mathbf{x}, t)$ is the activity field (characterizing the density of grains above threshold), $E(\mathbf{x}, t)$ is the locally-conserved energy field, a, b, ω, D, σ , and D_E are parameters and $\eta(\mathbf{x}, t)$ is a Gaussian white noise. Equation (1) represents a robust and well-established universality class including not only stochastic sandpiles, but also some reaction-diffusion systems [17, 21, 22].

In a recent Letter however, Mohanty and Dhar [23] have argued that generic sandpile models with “sticky grains” or “inertia”, where some grains remain stable even if the local threshold is passed, should be in the DP class. The authors presented convincing numerical and analytical evidence that indeed this is the case for a *directed* two-dimensional system, which happens to be mappable into an effective one-dimensional directed site-percolation dynamics. Also, for isotropic (*undirected*) models with stickiness it was claimed that DP scaling holds. For this second case the authors presented a Monte Carlo simulation and, additionally justified their findings by arguing that the “right” set of Langevin equations for inertial sandpiles should include a coupling of the form $\omega\rho\Theta(E - \rho - E_c)$ where Θ is the Heaviside step function and E_c is the instability threshold, substituting the bilinear coupling $\omega\rho E$ in Eq. (1).

The logic behind such a term is, in principle, reasonable [19] and it is argued in [23] that considering this coupling, one should leave the Manna universality class

and return to DP, i.e. the conservation law should be irrelevant in the presence of “stickiness” (see the schematic diagram in figure 4 of [23]).

In this paper, we argue that this claim in [23] is unfounded: even in the presence of inertia/stickiness, the generic universality class of (undirected) stochastic sandpiles models remains the C-DP/Manna one. The paper is organized as follows: we first report on extensive simulations of the model studied in [23] in the fixed-energy ensemble, from which we conclude that isotropic sticky sandpiles are in the C-DP/Manna class. In section III, we integrate the set of Langevin equations with a Θ function coupling, which we find to be also in the C-DP/Manna class, and we perform a numerical renormalization treatment to show that these Langevin equations evolve upon coarse-graining towards Eqs.(1).

II. MICROSCOPIC “STICKY” SANDPILE MODELS

The model proposed in [23] is a variation of the Manna model: a discrete sandpile, defined on a one-dimensional lattice, with a height threshold h_c , slow sand addition and dissipation at the boundaries, but including a (sticking) probability $1 - p$ for grains to remain stable even if they are above threshold. Here we consider only the limit which possesses a critical point, i.e. the bulk-dissipation rate is set to zero. Following the strategy in [13, 14, 15], we analyze the “fixed energy version” of the model: we suppress grain addition and boundary dissipation, fix the total energy to E , and use p as a control parameter. Note that, owing to the existence of a non-vanishing sticking probability, arbitrarily large heights are allowed. Active sites (at which $h > h_c$) are updated in parallel with the toppling occurring with probability p .

We have implemented two different versions in which each toppling event redistributes two grains to the two nearest-neighbor sites, either randomly (stochastic redistribution rule) or regularly, with one grain onto each neighbor (deterministic rule). The methodology followed is standard for absorbing phase transitions: first, the critical point is determined by studying the decay of activity from some initial condition varying p in a large system: for large p values, activity saturates (active phase), while for small p activity vanishes (absorbing phase). At the critical point, separating these two phases, p_c , activity decays asymptotically as a power-law with the critical exponent $\theta = \beta/\nu_\parallel$. For the stochastic and the deterministic rules, we find, respectively, $p_c = 0.84937(2)$ with $\theta = 0.120(8)$ and $p_c = 0.76750(3)$ with $\theta = 0.115(8)$ (Fig. 1a,b). These estimates of θ are in good agreement with the best evaluations for the Manna class in one dimension, *i.e.* $\theta = 0.125(2)$ [24], and clearly incompatible with the DP value $\theta \approx 0.159$.

Next, using the critical value determined above, the variation of the stationary saturation value of activity at the critical point for smaller system-sizes is recorded.

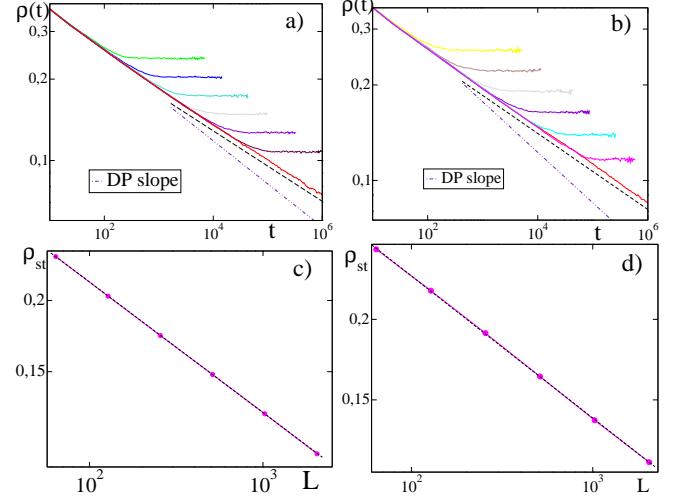


FIG. 1: (a-b) Log-log plot of the time-decay of the order parameter (activity density) for different system sizes (from top to bottom: $L = 64, 128, 256, 512, 1024, 2048$ and $L = 2^{18}$) and for the (a) stochastic rule and (b) the deterministic one, at their corresponding critical points $p_c = 0.84937(2)$ and $p_c = 0.76750(3)$. From the slopes, we determine $\theta = 0.120(8)$ and $\theta = 0.115(8)$ respectively (DP slopes are plotted for comparison). In (c-d) we plot the saturation values at the previously determined critical points for the stochastic (c) and deterministic (d) rules respectively. From the scaling at the critical point we determine $\frac{\beta}{\nu_\perp} = 0.22(1)$ for both of them.

From the expected scaling law $\rho_{st}(p = p_c) \sim L^{-\frac{\beta}{\nu_\perp}}$, we determine $\frac{\beta}{\nu_\perp} = 0.22(1)$ for both the stochastic and the deterministic rule as shown in Fig. 1c,d. Again, this value is in good agreement with available estimates for the C-DP/Manna class $\frac{\beta}{\nu_\perp} = 0.215(5)$ [25, 26], and incompatible with the DP value $\frac{\beta}{\nu_\perp} \approx 0.252$.

We have also performed spreading experiments [10, 12] (not shown) by following the standard procedure: we perturb a natural absorbing state (one generated by the system dynamics) to generate a small amount of localized activity and analyze how it spreads out at the previously determined critical point. We measured $\eta = 0.39(3)$, $\delta = 0.167(5)$, and $z = 1.39(3)$ exponents for the number of active sites, surviving probability, and average square-radius critical, respectively [10, 11, 12]. These values are in good agreement with the best estimations for the Manna class [17] and differ from their corresponding DP values ($\eta \approx 0.313$, $\delta \approx 0.159$, and $z \approx 1.258$ [18]).

III. NUMERICAL STUDY OF LANGEVIN EQUATIONS

After revisiting microscopic sandpile models with sticky grains, their critical behavior appears to be fully compatible with those of the C-DP/Manna class. We now turn to a study of the coupled Langevin equations

proposed by Mohanty and Dhar to describe their coarse-grained dynamics. The stochastic equations proposed in [23] are Eqs. 1, *i.e.* those of the C-DP/Manna class, except for the coupling term $\omega\rho E$ which is replaced by $\omega\rho\Theta(E - \rho - E_c)$ where E_c is the (microscopic) toppling threshold. The presence of this microscopic feature and of the step function Θ is surprising in so far as Langevin equations are usually understood as resulting from some coarse-graining of microscopic dynamics. In particular, the step function is unlikely to be a robust mesoscopic description, as it will be modified (probably transformed into a smoother function) upon coarse-graining.

Discontinuous functions are notoriously difficult in the framework of renormalization group analysis. Moreover, even the “simple” equations (1) resist standard perturbative renormalization attempts [27]. The only available strategy then is direct numerical integration. The presence of the (square-root) multiplicative noise term makes this a priori difficult [28], but this technical difficulty was recently circumvented by the fast and quasi-exact sampling method introduced in [24].

We have used this scheme to numerically integrate the equations proposed by Mohanty and Dhar. These simulations yielded two sets of results: following the protocol recalled above, we studied the absorbing phase transition observed when varying the linear coefficient. We also introduced a local effective “mass” coefficient

$$a_{\text{eff}}(\mathbf{x}, t) = a + \omega \Theta(E(\mathbf{x}, t) - \rho(\mathbf{x}, t) - E_c), \quad (2)$$

and studied its behavior upon coarse-graining in numerical simulations to analyze the relation between the two sets of Langevin equations.

A. Phase transition

Starting from an homogeneous, active, initial condition, we studied the time-decay of spatially-averaged activity varying the control parameter a . As expected, algebraic decay is found at the critical value separating exponential decay (absorbing phase) from saturation (active phase). The estimated decay exponent $\theta = 0.130(5)$ is in perfect agreement with the C-DP/Manna class value (Fig. 2a). At the critical point, $a = 0.72308$, the scaling of the stationary activity for finite size systems yields the estimate $\beta/\nu_{\perp} = 0.22(1)$, (Fig. 2b) again in agreement with the C-DP/Manna value. These estimates are thus incompatible with the DP class values. Also, spreading experiments (not shown) fully confirm this result.

B. Numerical coarse-graining

The above results are easily understood when observing the behavior of the Mohanty-Dhar Langevin equations coarse-grained numerically. To do this, we build scatter plots of $\langle a_{\text{eff}} \rangle_N$ vs the field difference $\langle E - \rho \rangle_N$,

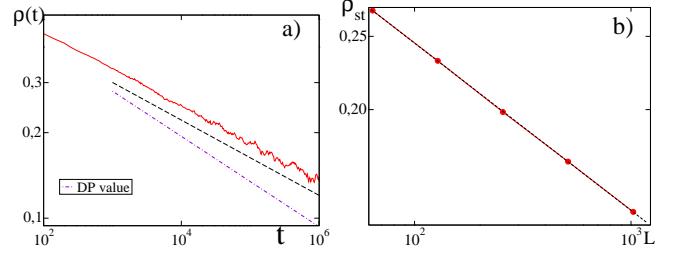


FIG. 2: Direct numerical integration of the Langevin equations proposed in [23]. (a) Decay experiments at the critical point $a = 0.72308$ (other parameter have been fixed as: system size $L = 2^{20}$, $\langle E(x, t = 0) \rangle = 0.5$, $b = 1$, $h_c = 0.5$, $D = D_E = 0.25$, $\omega^2 = \sigma^2 = 2$, integrated with time-step $dt = 0.25$). (b) Finite size scaling at criticality (more details in the text).

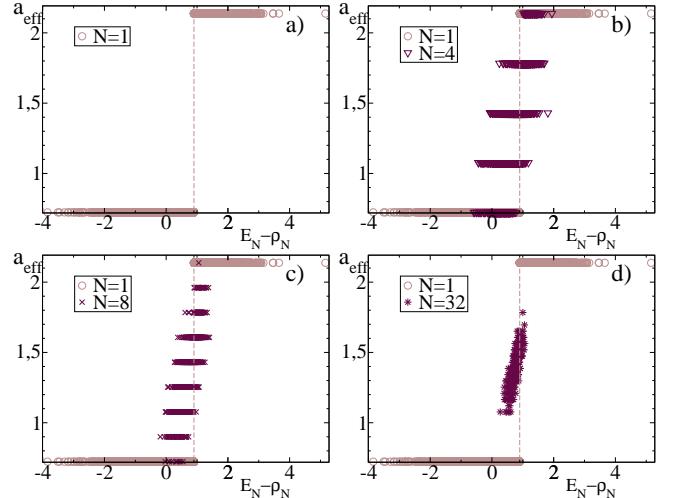


FIG. 3: Effective mass as defined by Eq. (2) as a function of the field difference averaged in Kadanoff blocks of size N , for $a = 0.72313$ in the active phase and $\omega = \sqrt{2}$. The vertical line corresponds to the threshold value $h_c = 0.9$. Observe that the larger the block size, the smoother the effective-mass dependence on the coarse-grained field difference.

where the averages are taken on Kadanoff-blocks of length N . For $N = 1$ (Fig. 3a), we obviously observe the Θ -function form. For $N = 4$ (Fig. 3b), the effective coupling can take intermediate discrete values between a and $a + \omega$, depending on the number of above-threshold microscopic sites in the block. When the block size is larger and larger (Fig. 3c,d), a smooth function gradually appears. By retaining just the two leading terms in a Taylor expansion of such an analytical function around the origin, we recover, at a coarse-grained level, the Langevin equation for the Manna class Eqs.(1), *i.e.* a linear coupling term and a correction to the linear term, a in Eqs.(1). Higher order terms in the Taylor expansion can be argued to be irrelevant from standard naive power count-

ing arguments. Therefore, it is not surprising that the set of Langevin equations including a Heaviside Θ function should exhibit the same asymptotic behavior as the original C-DP/Manna class Langevin equations, Eqs.(1).

IV. CONCLUSION AND DISCUSSION

To sum up, introducing “stickiness” in isotropic sandpile models does not change their universality class, which remains that of the Manna model. We reached this conclusion via numerical simulations of microscopic models and Langevin equations proposed in [23]. We showed in addition that these Langevin equations “flow” towards those of the Manna class under some numerical coarse-graining procedure.

One can wonder what is the reason why this conclusion does not hold for the *directed* (or anisotropic) sandpiles studied also by Mohanty and Dhar in [23] (see also

[29]), which they proved to be in the DP class. These directed models, defined on a two-dimensional lattice include an isotropic direction and a fully anisotropic one, in the sense that sand goes “downwards” in that direction but not “upwards”. This makes it possible to map the problem on DP in (1+1) dimensions, *i.e.* the anisotropic dimension can be taken as “time”. The local conservation of energy is present also in these models, but “local” here means in “space-time” neighborhoods, while *energy is not conserved* in the isotropic spatial direction. Hence, Manna behavior does not appear, while DP scaling, the usual one in the absence of spatial energy conservation, is expected to emerge, as indeed was proved in [23].

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